A Finite ε-convergence Algorithm for Two-stage Stochastic Convex Nonlinear Programs with Mixed-binary First and Second Stage Variables

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Stochastic Mixed Integer Programming (SMIP)

$$\min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d^T y_\omega$$

$$s.t. \quad Ax = b$$

$$W_\omega y_\omega = h_\omega - T_\omega x \quad \forall \omega \in \Omega$$

$$x \in \{x = (x1, x2) : x_1 \in \{0,1\}^n, \;\; x2 \geq 0\}$$

$$y_\omega \in \{y = (y1, y2) : y_1 \in \{0,1\}^m, \;\; y2 \geq 0\} \quad \forall \omega \in \Omega$$ **(Mixed) integer recourse**

Benders decomposition cannot be applied directly.

Convexified Subproblem:

$$\min \quad z_\omega = d^T y_\omega$$

$$s.t. \quad W_\omega y_\omega = h_\omega - T_\omega x^k$$

$$0 \leq y_\omega \leq y^{ub}$$

$$\alpha_{l\omega} x + \beta_{l\omega} y_\omega \leq \gamma_{l\omega} \quad \forall l \in L$$ **Cutting planes**

Stochastic Mixed Integer Programming (SMIP)

MILP:

Gomory cuts	Lift-and-project cuts	RLT	CPT
Gade et at., 2014	Caroe and Tind, 1998	Sherali and Fraticelli, 2002	Qi and Sen, 2017

Convex MINLP[1] (this talk):

- Represent the subproblems as **generalized disjunctive programs**
- Apply **basic steps** to the disjunctive programs to construct the **convex hull**
- **Spatial branch and bound** on the **continuous** first stage variables

[1]By convex MINLP, we mean mixed-integer nonlinear programs where the functions involved are convex, (see Lee and Leyffer 2011).

Convex MINLP with Mixed-integer Recourse

$$(P) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

Convex Nonlinear

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

$$x \in X, \quad X = \{x : x_i \in \{0,1\}, \forall i \in I_1, \ \ 0 \leq x \leq x^{ub}\}$$

$$y_\omega \in Y \quad \forall \omega \in \Omega, \quad Y = \{y : y_j \in \{0,1\}, \forall j \in J_1, \ \ 0 \leq y \leq y^{ub}\}$$

Mixed binary

Assumption 1. Both *x* and y_ω are bounded.
Assumption 2. The problem has relatively complete recourse.

Continuous Relaxation of the Second Stage

$$(PC) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x \in X$$

$$(x, y_\omega) \in conv(S_\omega) \quad \forall \omega \in \Omega$$

$$S_\omega = \left\{(x, y_\omega | A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega}, y_\omega \in Y, 0 \leq x \leq x^{ub}\right\}$$

How to obtain the convex hull of S_ω in closed-form?

Basics of Disjunctive Programming

$C_j = \{x \in \mathbb{R}^n | \phi_j(x) \leq 0\},\ j \in M \qquad \phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is a convex function

Union (elementary disjunctive set)

$$H = \bigcup_{j \in M} C_j = \{x \in \mathbb{R}^n | \bigvee_{j \in M} \phi_j(x) \leq 0\}$$

Intersection

$$P = \bigcap_{j \in M} C_j = \{x \in \mathbb{R}^n | \bigwedge_{j \in M} \phi_j(x) \leq 0\}$$

Conjunctive normal form intersection of some elementary disjunctive sets

$$F_{CNF} = \bigcap_{i \in T} H_i$$

Disjunctive normal form the union of convex sets

$$F_{DNF} = \bigcup_{i \in D} P_i$$

P_i is a convex set $P_i = \{x \in \mathbb{R}^n | g_i(x) \leq 0\}$, where $g_i(x)$: $\mathbb{R}^n \rightarrow \mathbb{R}^m$

Equivalent Convex Disjunctive Programs

Regular Form: Form represented by the intersection of the union of convex sets

$$F_{RF} = \bigcap_{k \in K} S_k, \; S_k = \cup_{i \in D_k} P_i$$

$$P_i \text{ a convex set } \forall i \in D_k$$

→ ***F is in regular form***

Theorem 1 *Let F_{RF} be a disjunctive set in regular form. Then F_{RF} can be brought to DNF by $|K| - 1$ recursive applications of the following basic step which preserves regularity:*

For some $r, s \in K$, bring $S_r \cap S_s$ to DNF by replacing it with:

$$S_{rs} = \bigcup_{i \in D_r, j \in D_s} (P_i \cap P_j)$$

Balas (1985)

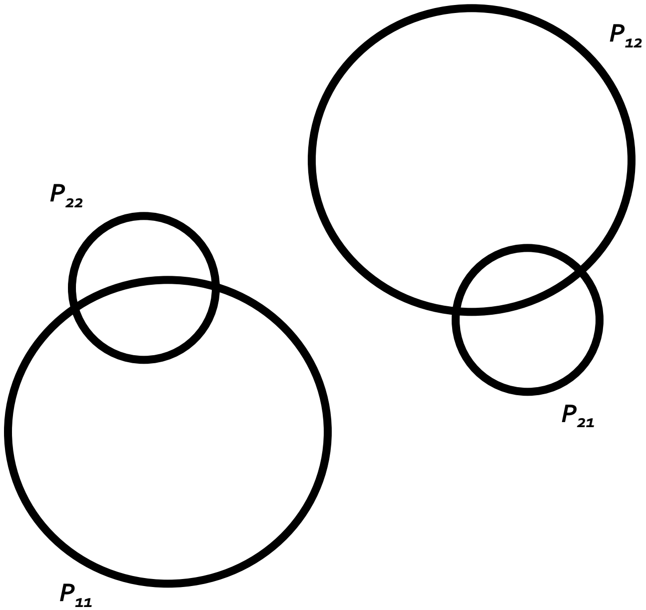
Hierarchy of Relaxations for Convex Disjunctive Programs

Theorem 2.4. *For $i = 1, 2....k$ let $F_i = \bigcap_{k \in K} S_k$ be a sequence of regular forms of a disjunctive set such that F_i is obtained from F_{i-1} by the application of a basic step, then:*

$$h\text{-}rel(F_i) \subseteq h\text{-}rel(F_{i-1})$$

(Ruiz, Grossmann, 2013)

Illustration: $F_0 = (P_{11} \cup P_{12}) \cap (P_{21} \cup P_{22})$



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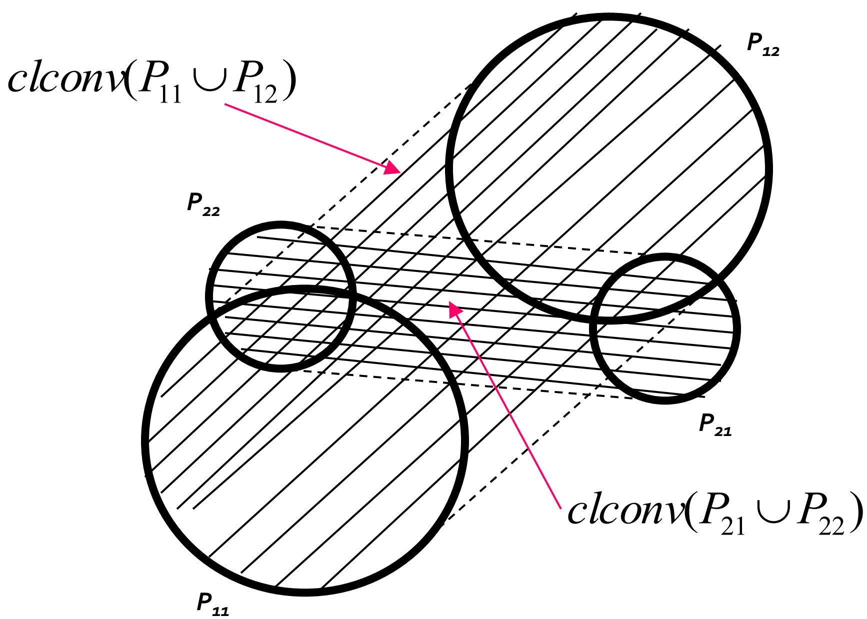

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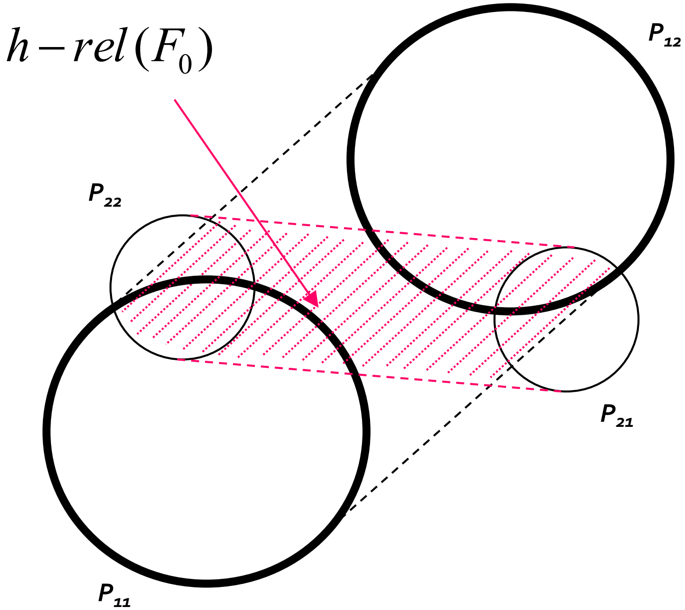

No Basic Step Applied => HR

Hierarchy of Relaxations for Convex Disjunctive Programs

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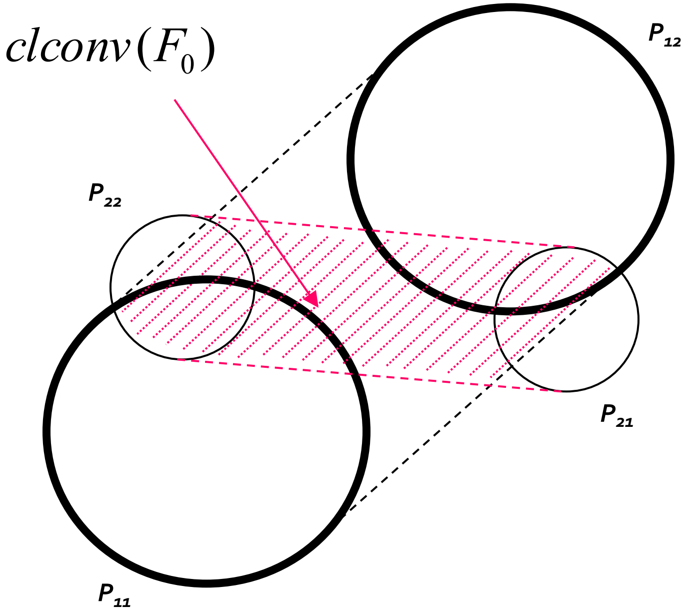
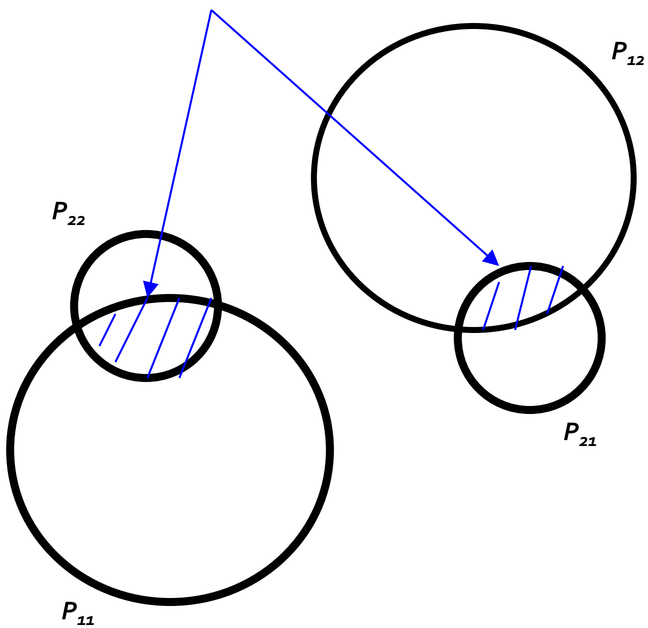
Illustration: $F_0 = (P_{11} \cup P_{12}) \cap (P_{21} \cup P_{22})$

$F_1 = (P_{11} \cap P_{21}) \cup (P_{11} \cap P_{22}) \cup (P_{12} \cap P_{21}) \cup (P_{21} \cap P_{22})$



No Basic Step Applied => HR



Basic Step Applied

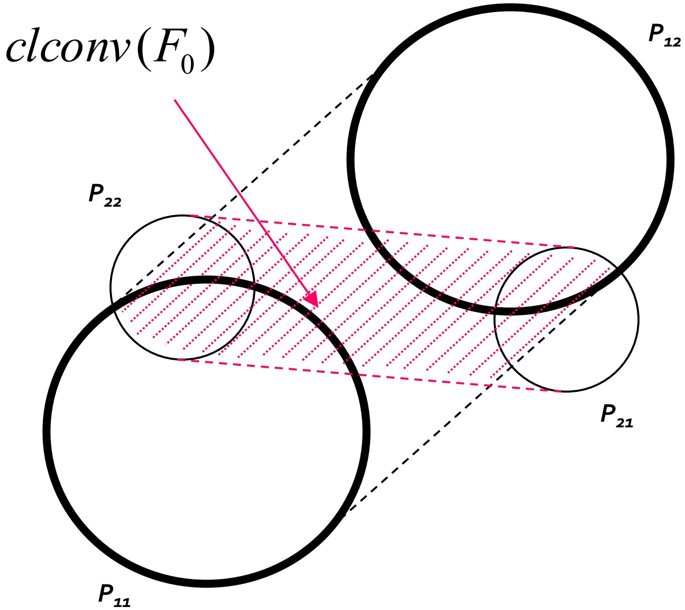
Hierarchy of Relaxations for Convex Disjunctive Programs

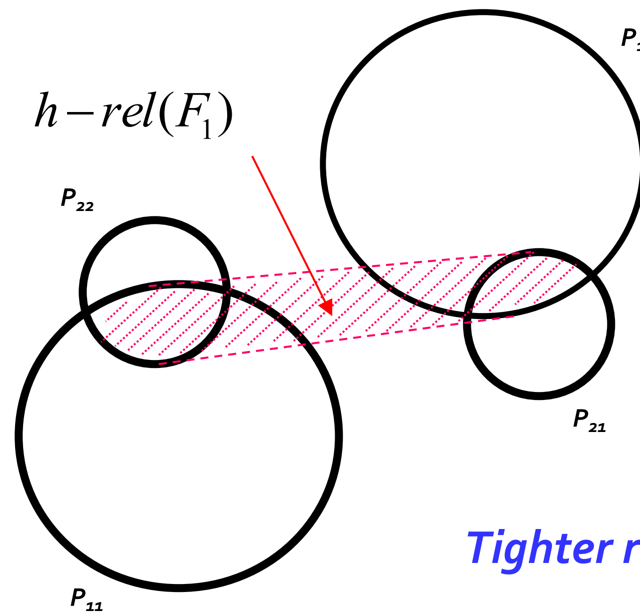
Theorem 2.4. *For $i = 1, 2....k$ let $F_i = \bigcap_{k \in K} S_k$ be a sequence of regular forms of a disjunctive set such that F_i is obtained from F_{i-1} by the application of a basic step, then:*

$$h\text{-}rel(F_i) \subseteq h\text{-}rel(F_{i-1})$$

(Ruiz, Grossmann, 2013)

Illustration: $F_0 = (P_{11} \cup P_{12}) \cap (P_{21} \cup P_{22})$ $\qquad F_1 = (P_{11} \cap P_{21}) \cup (P_{11} \cap P_{22}) \cup (P_{12} \cap P_{21}) \cup (P_{21} \cap P_{22})$





Tighter relaxation!

No Basic Step Applied => HR **Basic Step Applied => CH**

Convex Hull of the Second Stage Problem

$$S_\omega = \left\{(x, y_\omega | A_{1,\omega}x + g_{1,\omega}(y_\omega) \leq b_{1,\omega}, y_\omega \in Y, 0 \leq x \leq x^{ub}\right\}$$

$$Y = \left\{y : y_j \in \{0,1\}, \forall j \in J_1, \quad 0 \leq y \leq y^{ub}\right\}$$

Disjunctive set representation

$$\begin{bmatrix} A_{1,\omega}x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \\ 0 \leq x \leq x^{ub} \\ 0 \leq y_\omega \leq y^{ub} \\ (y_\omega)_j = 1 \end{bmatrix} \vee \begin{bmatrix} A_{1,\omega}x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \\ 0 \leq x \leq x^{ub} \\ 0 \leq y_\omega \leq y^{ub} \\ (y_\omega)_j = 0 \end{bmatrix} \quad \forall j \in J_1$$

Apply basic steps (intersection of disjunctions):

$$\vee_{r \in R} \begin{bmatrix} A_{1,\omega}x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \\ 0 \leq x \leq x^{ub} \\ 0 \leq y_\omega \leq y^{ub} \\ (y_\omega)_j = e_{rj} \quad \forall j \in J_1 \end{bmatrix}$$

(Balas, 1985)

(Ruiz, Grossmann, 2013)

Disjunctive Normal Form (DNF)

$2^{|J_1|}$ ***disjuncts***

set R all the possible combinations of the binary variables $(y_\omega)_j, \forall j \in J_1$

Convex Hull of the Second Stage Problem

$$x = \sum_{r \in R} u_\omega^r$$

$$y_\omega = \sum_{r \in R} v_\omega^r$$

$$\sum_{r \in R} \gamma_\omega^r = 1, \quad 0 \leq \gamma_\omega^r \leq 1, \quad \forall r \in R$$

$$A_{1,\omega} u_\omega^r + \gamma_\omega^r g_{1,\omega}(v_\omega^r / \gamma_\omega^r) \leq b_{1,\omega} \gamma_\omega^r, \quad \forall r \in R$$

$$0 \leq u_\omega^r \leq x^{ub} \gamma_\omega^r, \quad \forall r \in R$$

Perspective function

$$0 \leq v_\omega^r \leq y^{ub} \gamma_\omega^r, \quad \forall r \in R$$

$$(v_\omega)_j = e_{rj} \gamma_\omega^r \quad \forall j \in J_1, r \in R$$

Ceria and Soares (1999)

Equivalence of (P) and (PC)

$$(P) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

$$x \in X, \quad y_\omega \in Y \quad \forall \omega \in \Omega$$

$$(PC) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x \in X$$

$$(x, y_\omega) \in conv(S_\omega) \quad \forall \omega \in \Omega$$

$$S_\omega = \{(x, y_\omega | A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega},$$
$$y_\omega \in Y, 0 \leq x \leq x^{ub}\}$$

Can apply generalized Benders decomposition to solve (PC)

Are (PC) and (P) equivalent?

Not in general. But there are some exceptions.

Pure Binary First Stage

$$(P) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

$$x \in X, \quad y_\omega \in Y \quad \forall \omega \in \Omega$$

$$(PC) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x \in X$$

$$(x, y_\omega) \in conv(S_\omega) \quad \forall \omega \in \Omega$$

$$S_\omega = \{(x, y_\omega | A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega}, \\ y_\omega \in Y, 0 \leq x \leq x^{ub}\}$$

Proposition 1 *Consider a special case of (P) where the first stage variables are all binary. We assume that in the corresponding problem (PC), the convex hull of S_ω is expressed in closed-form. Then (P) and (PC) are equivalent in the sense that they have the same optimal objective value and the optimal solution of (P) can always be obtained based on the optimal solution of (PC).*

Mixed Binary First Stage

$$(P) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

$$x \in X, \quad y_\omega \in Y \quad \forall \omega \in \Omega$$

$$(PC) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x \in X$$

$$(x, y_\omega) \in conv(S_\omega) \quad \forall \omega \in \Omega$$

$$S_\omega = \{(x, y_\omega | A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega}, \\ y_\omega \in Y, 0 \leq x \leq x^{ub}\}$$

Corollary 1 *For (PC) with both binary and continuous first stage variables, if the optimal first stage variables x^* to (PC) are all at their upper or lower bound, i.e., $(x^*)_i = 0$ or $(x^*)_i = (x^{ub})_i$, $\forall i \in I$, then (PC) and its corresponding (P) are equivalent in the sense that they have the same optimal objective value.*

Spatial Branch and Bound

Spatial: Branch on continuous variables

Problem at node *q*

$$(PCBAB_q) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x_q^{lb} \leq x \leq x_q^{ub}$$

$$x \in X$$

$$(x, y_\omega) \in conv(S_\omega^q)$$

$$S_\omega^q = \{(x, y_\omega | A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega}, y_\omega \in Y, x_q^{lb} \leq x \leq x_q^{ub}\}$$

Branch on continuous *x* to satisfy the condition for Corollary 1

Branching rule: branch on the variable whose optimal value has largest distance to its bounds

Convergence of GBDBAB

GBDBAB

Generalized Benders Decomposition-based Branch and Bound

1. Initialization

$UB = +\infty,\ LB = -\infty,\ k = 1,\ L_k = \{q^0\},\ x^{lb}_{q^0} = 0,\ x^{ub}_{q^0} = x^{ub}$

Solve $(PCBAB_{q^0})$. Update LB, UB

2. Branching

Select node q from L_k. Create two new nodes q_1, q_2

Add branching constraints to q_1, q_2. Delete q from L_k

Add q_1, q_2 to L_k. $k = k + 1$

3. Bounding

Solve $(PCBAB_{q_1})$, $(PCBAB_{q_2})$ with GBD.

Update UB, LB.

4. Fathoming

Fathom by optimality if $V(PCBAB_q) \leq v(PCBAB_q) + \epsilon$

Fathom by bound if $v(PCBAB_q) + \epsilon \geq UB$

Check if $L_k = \emptyset$

no → 2. Branching

yes → Stop

Proposition 2 *The algorithm GBDBAB has finite ϵ-convergence.*

A Sequential Convexification Scheme

Disjunctive set representation

$$\begin{bmatrix} A_{1,\omega}x + g_{1,\omega}(y_\omega) \le b_{1,\omega} \\ x_q^{lb} \le x \le x_q^{ub} \\ 0 \le y_\omega \le y^{ub} \\ (y_\omega)_j = 1 \end{bmatrix} \vee \begin{bmatrix} A_{1,\omega}x + g_{1,\omega}(y_\omega) \le b_{1,\omega} \\ x_q^{lb} \le x \le x_q^{ub} \\ 0 \le y_\omega \le y^{ub} \\ (y_\omega)_j = 0 \end{bmatrix} \quad \forall j \in J_1$$

$S_{\omega j}^{q1} \cup S_{\omega j}^{q0},\ j \in J_1$

$\cap_{j \in J_1} (S_{\omega j}^{q1} \cup S_{\omega j}^{q0})$

Partial application of basic steps (intersection of disjunctions):

$$S_\omega^q = \cap_{t \in T_\omega^q} \big(\cap_{j \in D_{\omega t}^q} \ (S_{\omega j}^{q1} \cup S_{\omega j}^{q0}) \big)$$

$$\cap_{j \in D_{\omega t}^q} (S_{\omega j}^{q1} \cup S_{\omega j}^{q0}) = \cup_{r \in R_{\omega t}^q} S_{\omega tr}^q$$

Use the hull relaxation of: $\cap_{t \in T_\omega^q} (\cup_{r \in R_{\omega t}^q} S_{\omega tr}^q)$

A Sequential Convexification Scheme

$$(PCBAB_q^l) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$x_q^{lb} \leq x \leq x_q^{ub}$$

$$(x, y_\omega) \in h - rel(S_{\omega l}^q), \quad \forall \omega \in \Omega$$

The hull relaxation after applying *l* basic steps

When should we stop?

Stop if the optimal solution is already in the convex hull of S_ω^q

Proposition 3 *For a given scenario ω, if $\exists t' \in T_\omega^q$ such that $(v_{\omega t'}^{r*}/\gamma_{\omega t'}^{r*})_j$, $\forall r \in R_{\omega t'}^q, \gamma_{\omega t'}^{r*} > 0, j \in J_1$, are 0 or 1, i.e., $v_{\omega t'}^{r*}/\gamma_{\omega t'}^{r*}$ satisfy the integrality constraints in S_ω^q, we have $(x_l^{q*}, y_{l\omega}^{q*}) \in conv(S_\omega^q)$.*

An Illustrative Example

$$\min \quad x_1 + x_2 + 3x_3 + 3x_4 + \sum_{\omega=\omega_1,\omega 2} \tau_\omega \left(y_{1\omega} - 12y_{2\omega} + 100y_{3\omega} + 3y_{4\omega} - 3y_{5\omega} \right)$$

$$x_1 \leq 4x_3, \quad x_2 \leq 2x_4,$$

$$x_1, x_2 \geq 0 \quad x_3, x_4 \in \{0,1\}$$

$$y_{1\omega} \leq x_1, \quad y_{2\omega} \leq x_2 \quad \forall \omega = \omega_1, \omega_2$$

$$(y_{1\omega} - 3)^2 + (y_{2\omega} - 2)^2 \leq 1 + 16(1 - y_{4\omega}) \quad \forall \omega = \omega_1, \omega_2$$

$$(y_{1\omega} - 1)^2 + y_{2\omega}^2 \leq 1 + 16y_{4\omega} \quad \forall \omega = \omega_1, \omega_2$$

$$y_{1\omega}^2 + (y_{2\omega} - 1)^2 \leq 1 + 16(1 - y_{5\omega}) \quad \forall \omega = \omega_1, \omega_2$$

$$(y_{1\omega} - 4)^2 + (y_{2\omega} - 1)^2 \leq 1 + 16y_{5\omega} \quad \forall \omega = \omega_1, \omega_2$$

$$y_{1\omega} + y_{2\omega} + y_{3\omega} \geq d_\omega \quad \forall \omega = \omega_1, \omega_2$$

$$y_{1\omega}, y_{2\omega}, y_{3\omega} \geq 0, \quad y_{4\omega}, y_{5\omega} \in \{0,1\} \quad \forall \omega = \omega_1, \omega_2$$

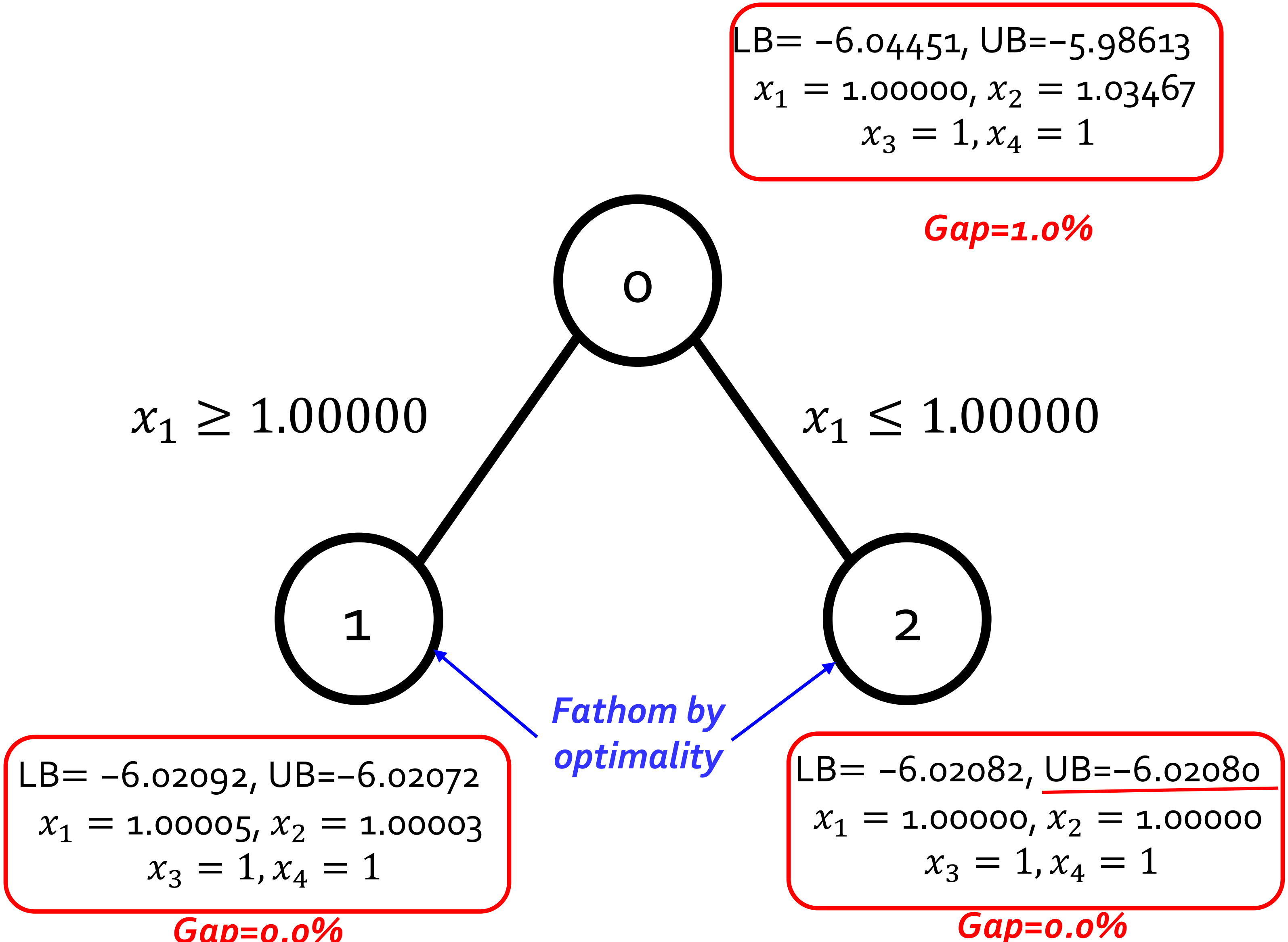
Convex nonlinear constraints

Uncertainty

Mixed-binary variables

Optimal value from DICOPT: -6.02080

An Illustrative Example



Computational Results of the Illustrative Example

Deterministic Equivalent

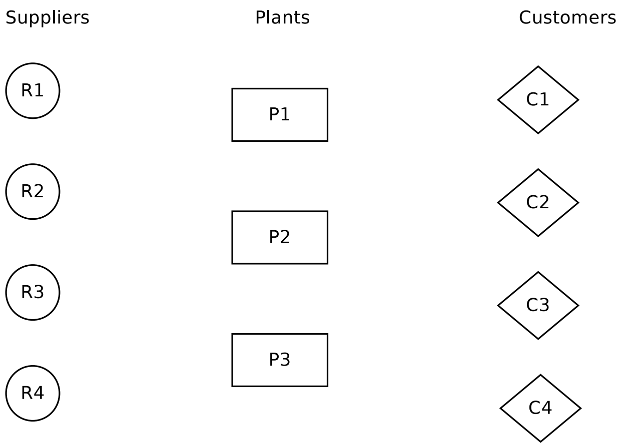
Scenarios	Linear Constr	Nonlinear Constr	Binary Var	Continuous Var	SBB s(gap)	AlphaECP s(gap)	DICOPT s(gap)
20	62	80	42	62	Timed out(7%)	9	2
60	182	240	122	182	Timed out(234%)	60	10
150	452	600	302	452	Timed out(245%)	254	685
300	902	1200	602	902	Timed out(247%)	917	Timed out(9%)

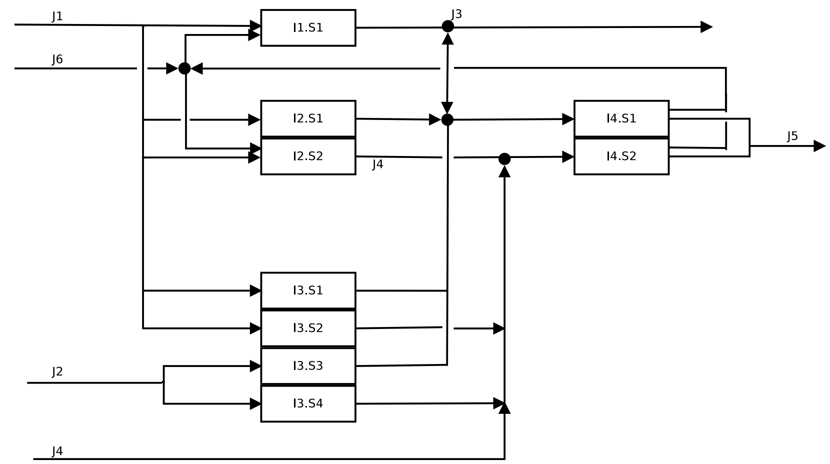
Proposed GBDBAB

Faster for large problems

Scenarios	Time (s)	Master (s)	Subproblem (s)	UB subproblem (s)	Nodes	Max Iterations
20	80	3	59	10	3	21
60	76	3	58	5	3	23
150	111	10	81	9	3	25
300	121	11	81	11	3	22

Planning under Demand and Price Uncertainty





- **First-stage decisions**
  - **Binary variables:** which process to install in each plant
  - **Continuous variables:** the capacity of each installed process
- **Second-stage decisions**
  - **Binary variables:** whether the transportation links are built
  - **Continuous variables:** purchase amount of raw materials, etc.
- **Constraints:** satisfy demands, production rate constraints, etc.
- **Objective:** minimize expected total cost

Planning Problem under Uncertainty

Problem from Li and Grossmann (2018). 2 suppliers, 2 plants, 2 customers.

Deterministic Equivalent

Scenarios	Linear Constr	Nonlinear Constr	Binary Var	Continuous Var	AlphaECP s(gap)	SBB s(gap)	DICOPT s(gap)
3	332	12	32	338	6	49	3
9	980	36	80	998	81	Timed out(2%)	9
27	2,924	108	224	2,978	3530	Timed out(19%)	96
81	8,756	324	656	8,918	Timed out (2%)	Timed out(40%)	Timed out(0.2%)

GBDBAB

All the problems solved at root node

Scenarios	Time (s)	Master (s)	Subproblem (s)	UB subproblem (s)	Basic step (max, min)
3	705	59	491	16	(7,2)
9	1,221	92	861	23	(5,1)
27	1,859	216	1,403	17	(0,0)
81	7,994	1,534	5,091	83	(1,0)

Conclusion

- We have proposed a **generalized Benders decomposition-based branch and bound algorithm** for two-stage convex 0-1 mixed-integer nonlinear stochastic programs with mixed-integer first and second stage variables
- **Sequential convexification** could sometimes help avoid the exponential representation of the **convex hull**

Acknowledgment

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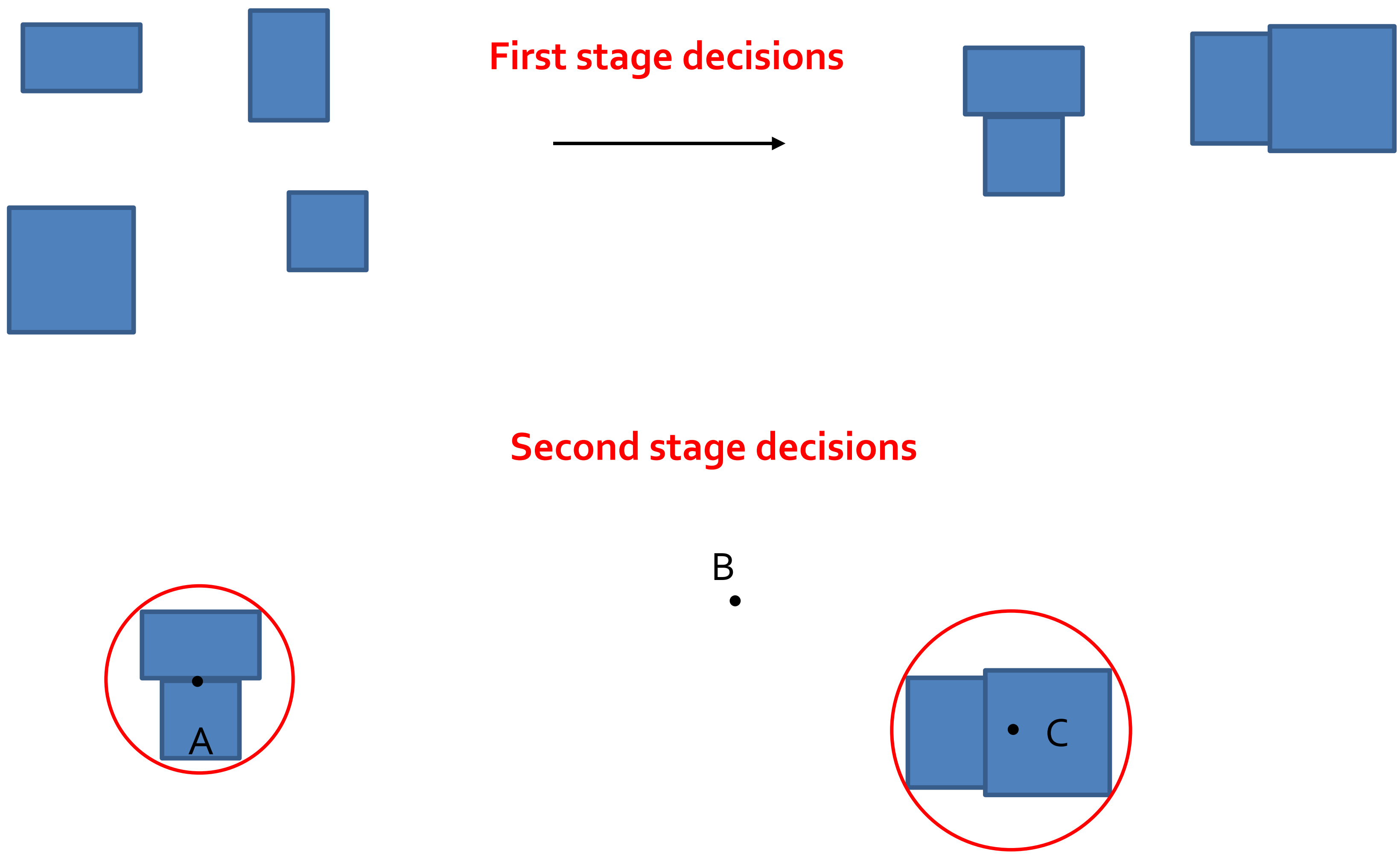
Questions?

Constrained Layout Problem under Price Uncertainty



Constrained Layout Problem under Price Uncertainty

Deterministic Equivalent (big-M reformulation)

Scenarios	Linear Constr	Nonlinear Constr	Binary Var	Continuous Var	SBB s(gap)	AlphaECP (s)	DICOPT s(gap)
3	54	72	30	36	14	120	3
9	108	216	66	84	4142	1478	43
36	351	864	228	300	Timed out(94%)	Timed out	Timed out(84%)
100	927	2400	612	812	Timed out(98%)	Timed out	Timed out(97%)

Deterministic Equivalent (hull reformulation)

Scenarios	Linear Constr	Nonlinear Constr	Binary Var	Continuous Var	SBB s(gap)	AlphaECP (s)	DICOPT s(gap)
3	360	72	30	180	38	32	6
9	702	216	66	372	5626	638	49
36	2241	864	228	1236	Timed out(52%)	Timed out (46%)	Timed out
100	5889	2400	612	3284	Timed out(82%)	Timed out (67%)	Timed out

Constrained Layout Problem under Price Uncertainty

GBDBAB

Scenarios	Time (s)	Master (s)	Subproblem (s)	UB subproblem (s)	Nodes
3	137	40	63	8	1
9	276	65	17	17	3
36	444	117	277	20	1
100	537	85	363	39	1