On Solving Nonconvex Two-Stage Stochastic Programs with Generalized Benders Decomposition

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Stochastic Mixed Integer Programming (SMIP)

$$\min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d^T y_\omega$$

$$s.t. \quad Ax = b$$

$$W_\omega y_\omega = h_\omega - T_\omega x \quad \forall \omega \in \Omega$$

$$x \in X, \quad X = \left\{ x : x_i \in \{0,1\}, \forall i \in I_1, \ \ 0 \le x \le x^{ub} \right\}$$

(Mixed) integer recourse

$$y_\omega \in Y \quad \forall \omega \in \Omega, \quad Y = \left\{ y : y_j \in \{0,1\}, \forall j \in J_1, \ \ 0 \le y \le y^{ub} \right\}$$

Benders decomposition cannot be applied directly.

In order to apply Benders decomposition

Try to solve subproblem:

$$\min \quad d^T y_\omega$$

$$s.t. \quad W_\omega y_\omega = h_\omega - T_\omega x^k \quad \forall \omega \in \Omega$$

$$y_\omega \in Y \quad Y = \left\{ y : y_j \in \{0,1\}, \forall j \in J_1, \ \ 0 \le y \le y^{ub} \right\}$$

No strong duality in the subproblem

Stochastic Mixed Integer Programming (SMIP)

In order to apply Benders decomposition

Convexified Subproblem:

$$\begin{aligned} \min \quad & z_\omega = d^T y_\omega \\ s.t. \quad & W_\omega y_\omega = h_\omega - T_\omega x^k \\ & 0 \le y_\omega \le y^{ub} \\ & \alpha_{l\omega} x + \beta_{l\omega} y_\omega \le \gamma_{l\omega} \quad \forall l \in L \end{aligned}$$

Cutting planes

Cuts valid for set $S_\omega = \{(x, y_\omega) | W_\omega y_\omega = h_\omega - T_\omega x, y_\omega \in Y, 0 \le x \le x^{ub}\}$

MILP:

Gomory cuts	Lift-and-project cuts	RLT	CPT
Gade et at., 2014	Caroe and Tind, 1998	Sherali and Fraticelli, 2002	Qi and Sen, 2017

Few works have been reported for MINLP

RLT: Reformulation Linearization Technique   CPT: Cutting Plane Tree

Stochastic Mixed Integer Nonlinear Programming

This talk

$$(P) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

Nonconvex Nonlinear

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

$$x \in X, \quad X = \{x : x_i \in \{0,1\}, \forall i \in I_1, \ \ 0 \leq x \leq x^{ub}\}$$

Mixed binary

$$y_\omega \in Y \quad \forall \omega \in \Omega, \quad Y = \{y : y_j \in \{0,1\}, \forall j \in J_1, \ \ 0 \leq y \leq y^{ub}\}$$

Assumption 1 *Problem (P) has relatively complete recourse, i.e., any solution x that satisfies the first stage constraints has feasible recourse decisions in the second stage.*

Assumption 2 *The feasible region of (P) is compact.*

Assumption 3 *x^{ub} and y^{ub} are finite, i.e., both the first and the second stage decisions are bounded.*

Previous Work

- **Pure binary first stage variables**
  - Nonconvex Generalized Benders Decomposition (Li et al., 2011)
- **Mixed-integer first stage variables**
  - A joint decomposition algorithm that has **convex** Benders subproblems (Ogbe and Li, 2018)
  - Perfect Information-based **branch and bound** (Cao and Zavala, 2017)
  - Modified Lagrangean decomposition-based **branch and bound** (Kannan and Barton, 2018)

Branch and bound tends to have slow convergence computationally

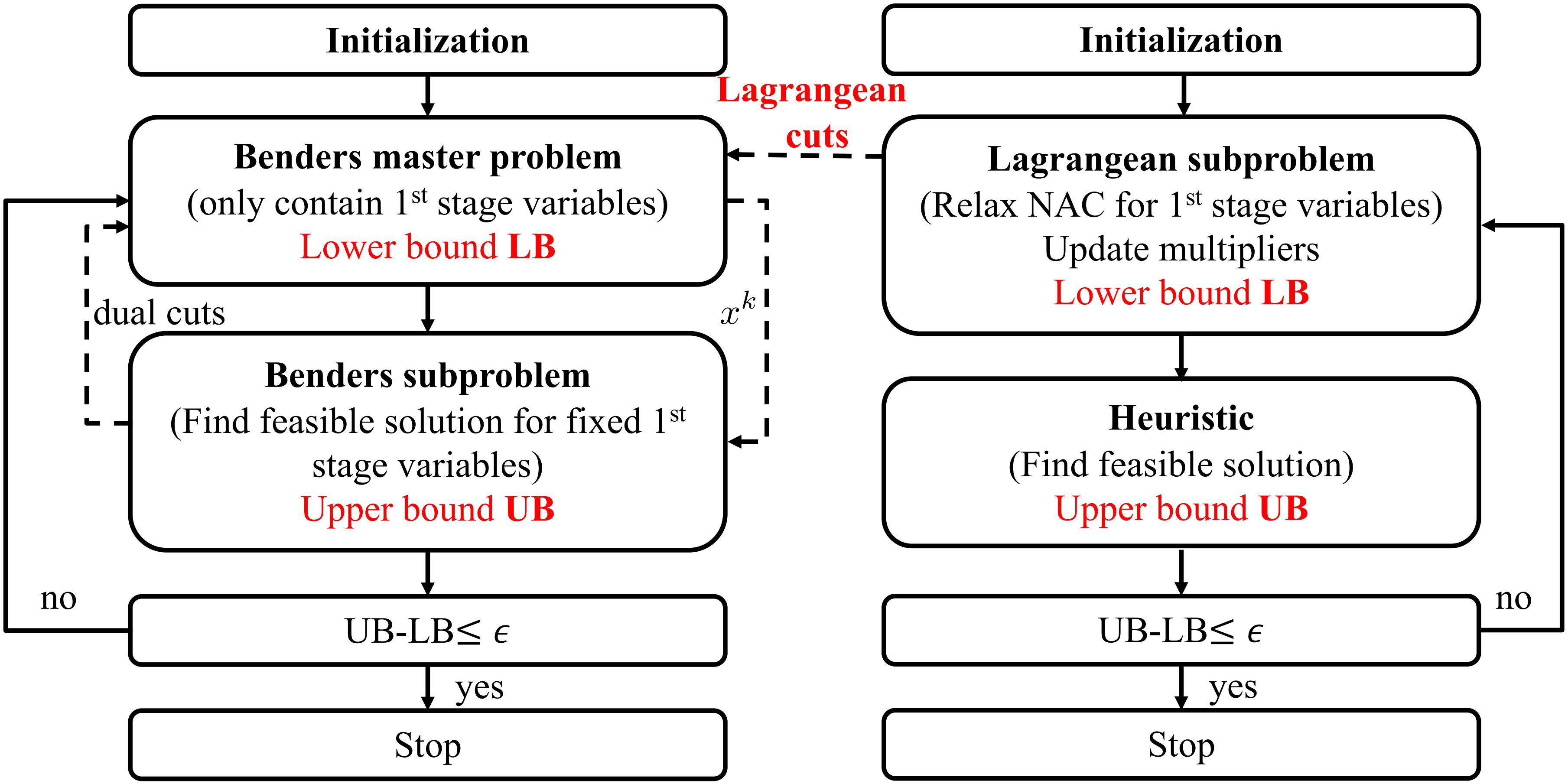
Branch and cut

Solution Strategy

- At high level: **Spatial branch and bound** to solve the problem to optimality
  - Branch on the **first stage variables**
- At low level: Each node in the BAB process is solved by **Generalized Benders-like** decomposition algorithm with **cutting planes**
- Two types of **valid inequalities** in the Benders **master** problem
- **Lagrangean cuts**
  - Combine with Lagrangean decomposition
- **Benders cuts**
  - **Convexify** the Benders subproblem with **cutting planes**

Lagrangean cuts

Benders decomposition | **Lagrangean decomposition**



$$\eta_\omega \geq z_{SL,\omega}^{*k} - \mu_\omega^k x \qquad \forall \omega \in \Omega, k \in K$$

Mitra et al. 2016

Lagrangean cuts

Proposition 1 $\eta_\omega \geq z^{*k}_{SL,\omega} - \mu^k_\omega x$ *is valid for the Benders master problem.*

Proposition 2 *The Benders master problem with the Lagrangean cuts yields a lower bound that is at least as tight as using Lagrangean decomposition.*

Li and Grossmann (2018)

Benders cuts

$$\begin{aligned} \min \quad & \tau_\omega d^T y_\omega \\ s.t. \quad & x = \tilde{x}^k \\ & A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \\ & y_\omega \in \{y : y_j \in \{0,1\}, \forall j \in J_1, \;\; 0 \leq y \leq y^{ub}\} \end{aligned}$$

Nonconvex

$$\begin{aligned} \min \quad & \tau_\omega d^T y_\omega \\ s.t. \quad & x = \tilde{x}^k \\ & A_{1,\omega} x + \tilde{g}_{1,\omega}(y_\omega, t_\omega) \leq b_{1,\omega} \\ & y_\omega \in \{y : y_j \in \{0,1\}, \forall j \in J_1, \;\; 0 \leq y \leq y^{ub}\} \end{aligned}$$

Convex relaxation

Mixed-integer

$$\begin{aligned} (SB_\omega^k): \quad \min \quad & z_{SB,\omega}^k = \tau_\omega d^T y_\omega \\ s.t. \quad & x = \tilde{x}^k \\ & A_{1,\omega} x + \tilde{g}_{1,\omega}(y_\omega, t_\omega) \leq b_{1,\omega} \\ & \alpha_{kj}^x x + \alpha_{kj}^y y_\omega + \alpha_{kj}^u t_\omega \geq \beta_{kj} \quad \forall k \in K, j \in J_1^k \\ & 0 \leq y \leq y^{ub} \end{aligned}$$

Cutting planes

Now the subproblem is continous and convex

Benders Master Problem

$$(MB^k): \quad \min \quad z_{MB}^k = \sum_{\omega} \eta_\omega$$

$$s.t. \quad A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$\eta_\omega \geq z_{SL,\omega}^{*l} - \mu_\omega^l x \quad \forall \omega \in \Omega, l \in L$$

Lagrangean cuts

$$\eta_\omega \geq z_{SB,\omega}^{*k} + (\lambda_\omega^k)^T (x - \tilde{x}^k) + \tau_\omega c^T x \quad \forall \omega \in \Omega, k \in K$$

Benders cuts

$$x \in X, \quad X = \left\{ x : x_i \in \{0,1\}, \forall i \in I_1, \ 0 \leq x \leq x^{ub} \right\}$$

Still has duality gap. Need to do spatial branch and bound.

GBD-based Branch and Cut

Problem Solved at node *q*

$$(P_q) \quad \min \quad c^T x + \sum_{\omega \in \Omega} \tau_\omega d_\omega^T y_\omega$$

$$A_0 x \geq b_0, \quad g_0(x) \leq 0$$

$$A_{1,\omega} x + g_{1,\omega}(y_\omega) \leq b_{1,\omega} \quad \forall \omega \in \Omega$$

Branch on stage 1 variables.

$$x \in X_q, \quad X_q = \{x : x_i \in \{0,1\}, \forall i \in I_1, \ \boxed{x_q^{lb} \leq x \leq x_q^{ub}}\}$$

$$y_\omega \in Y \quad \forall \omega \in \Omega, \quad Y = \{y : y_j \in \{0,1\}, \forall j \in J_1, \ 0 \leq y \leq y^{ub}\}$$

We can solve each (P_q) by generalized Benders decomposition with Lagrangean cuts.

Node Selection & Branching Rules

Node Selection rule

$$\text{Select node } q \text{ such that } q = \arg\min_{q \in \Gamma} LB_q.$$

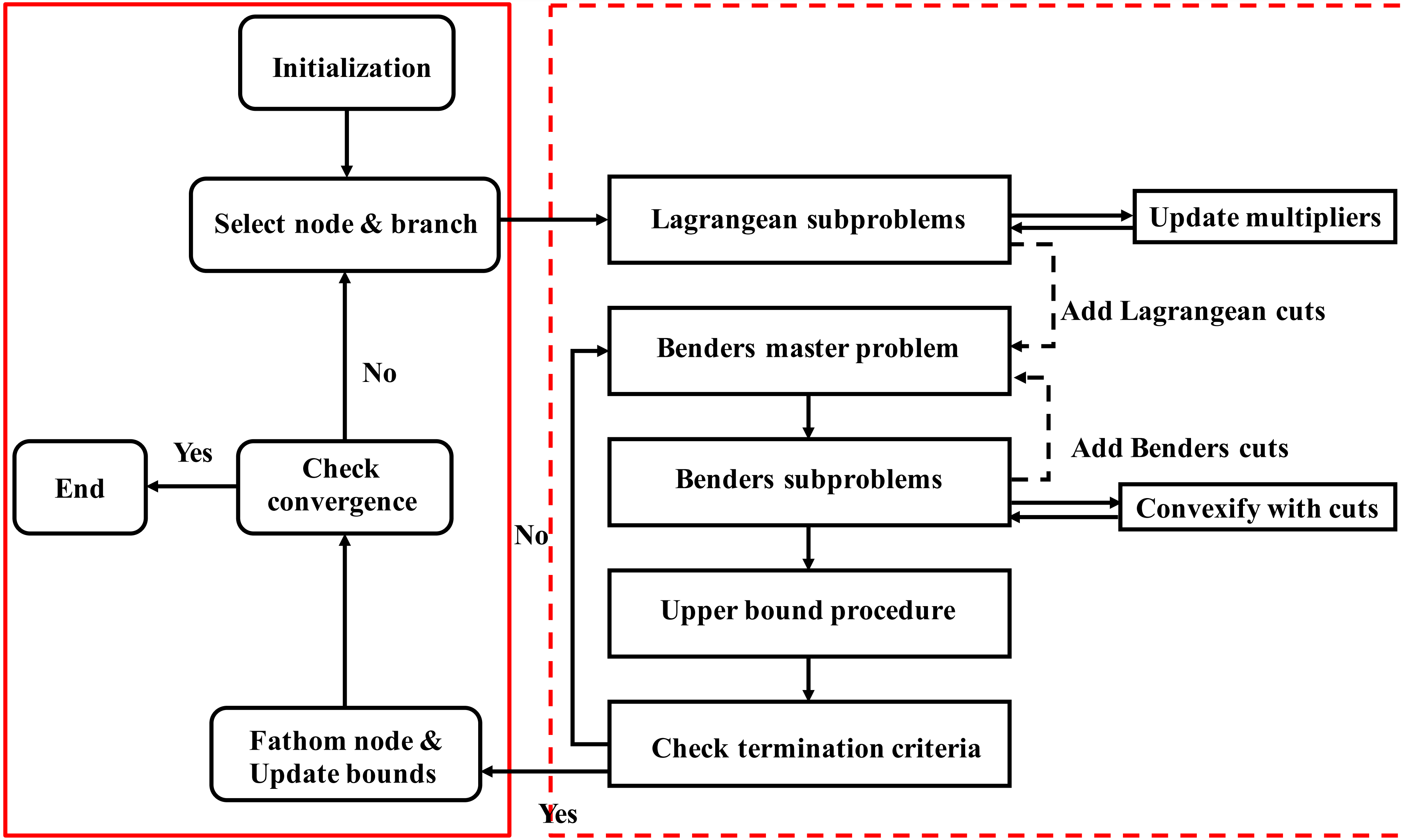
Branching rule example

Select the first stage variable with the largest normalized relative diameter,

$$i^* = \underset{i \in I}{\text{argmax}} \frac{(x_q^{ub})_i - (x_q^{lb})_i}{(x_{q_0}^{ub})_i - (x_{q_0}^{lb})_i} \delta_i$$

q_0 represents the root node. δ_i is a normalization factor for variable i. Two new nodes q_1 and q_2 are then created and bisect the domain of variable i^*.

GBD-based Branch and Cut



Spatial branch and bound **Solve a single node**

Definition

Definition 1 *A subdivision is called exhaustive if* $\lim_{p\to\infty} \delta(X_{q_p}) = 0$*, for all decreasing subsequences* X_{q_p} *generated by the subdivision.*

Definition 2 *A selection operation is said to be bound improving if, after a finite number of steps, at least one partition element where the actual lower bounding is attained is selected for further partition.*

Definition 3 *The "deletion by infeasibility" rule throughout a branch and bound procedure is called certain in the limit if, for every infinite decreasing sequence* $\{X_{q_p}\}$ *of successively refined partition elements with limit* $\bar{X}$*, we have* $\bar{X} \cap D \neq \emptyset$*.*

Definition 4 *A lower bounding operation is called strongly consistent if, at every iteration, any undeleted partition set can be further refined and if any infinite decreasing sequence* $\{X_{q_p}\}$ *successively refined partition elements contains a sub-sequence* $\{X_{q_{p'}}\}$ *satisfying* $\bar{X} \cap D \neq \emptyset$*,* $\lim_{p\to\infty} LB_{q_p} = z^*(\bar{X} \cap D)$*, where* $\bar{X} = \cap_p X_{q_p}$*.*

Horst and Tuy (2013)

Convergence

Lemma 1 *The subdivision process of the proposed algorithm is exhaustive.*

Lemma 2 *The selection operation of the proposed algorithm is bound improving.*

Lemma 3 *Deletion by infeasibility is certain in the limit in the proposed algorithm.*

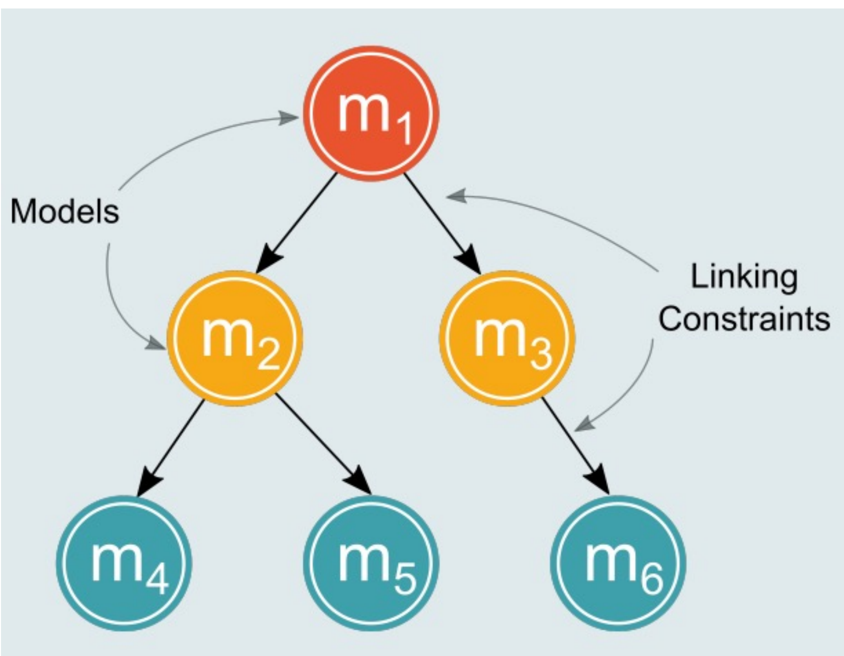
Lemma 4 *The proposed algorithm is strongly consistent.*

Theorem 1 *The proposed algorithm is convergent, i.e.,* $\lim_{q \to \infty} LB_q = \lim_{q \to \infty} UB_q = z^*$.

Horst and Tuy (2013)

Implementation

- Implemented using **Plasmo** v0.0.1 (Jalving et al., 2017) in **JuMP Julia.**
- **PlasmoAlgorithm.jl:** Julia package that implements **decomposition algorithms** using Plasmo graphs as input.



Applications of the Algorithm

- **Stochastic Pooling Problem with Contrast Selection**
  Stochastic Pooling Problem with Contract Selection
- Crude Selection and Refinery Optimization Under Uncertainty (Yang and Barton, 2016)
- Storage Design for a Multi-product Plant under Uncertainty (Rebennack et al. 2011)

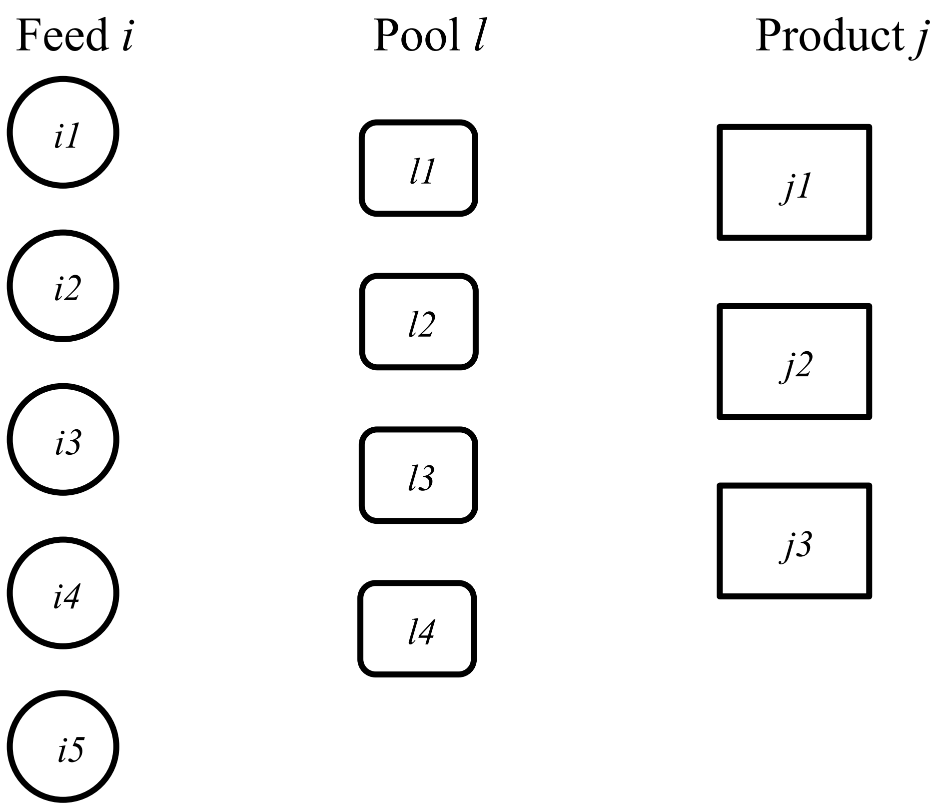
Stochastic Pooling Problem with Contract Selection

- **Stage 1 decisions:** feeds and pools selection (binary), feeds and pools capacity(continuous).
- **Stage 2 decisions:** contract selection for feeds (binary), amount of feeds purchased under each contract, mass flow rate (continuous).
- **Constraints:** capacity limitation, mass balance, quality specifications (bilinear), contract selection.
- **Sources of Uncertainty:** Demand of products. Price of feeds. Selling price of products.

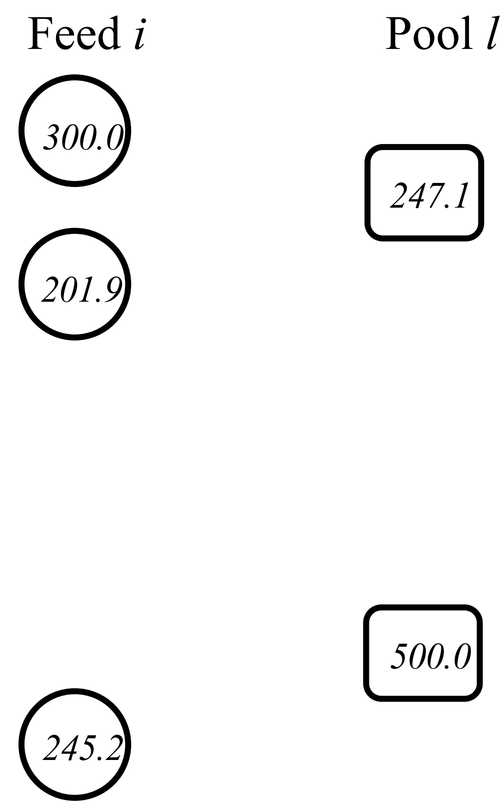
Stochastic Pooling Problem with Contract Selection

Case study – 3 scenario problem

Possible structure



Stage 1 decisions



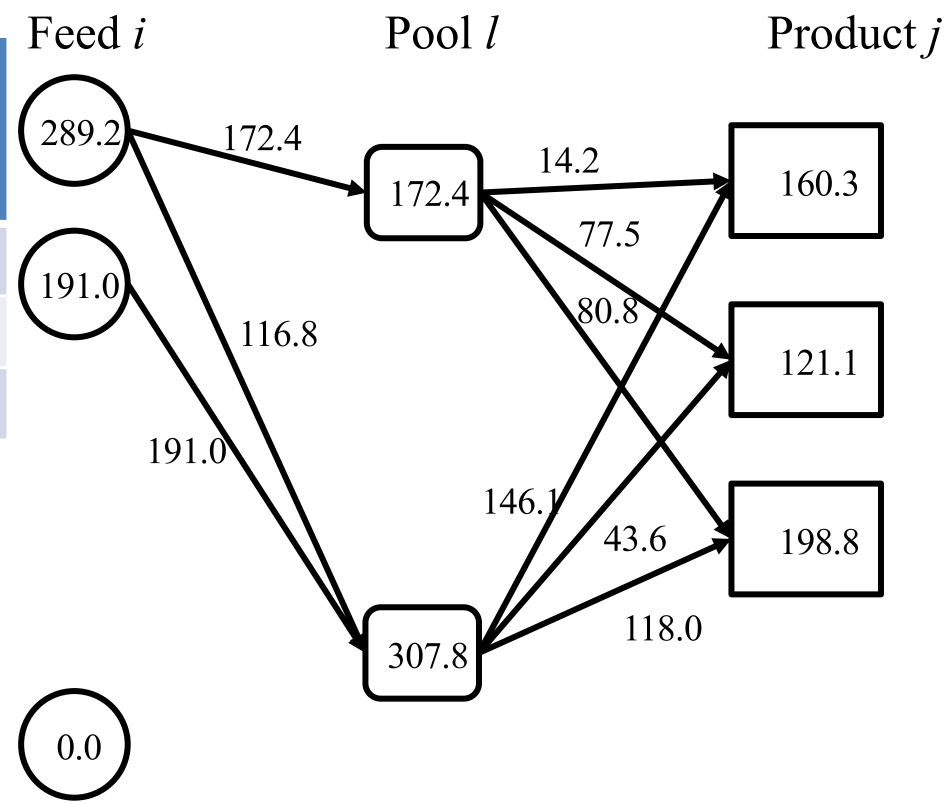
Cost = 2414.98

Stochastic Pooling Problem with Contract Selection

Case study – 3 scenario problem

Second stage decisions—low demand

Contracts	fixed	Discount after certain amount	Bulk discount
Feed 1		289.2	
Feed 2			191.0
Feed 5			



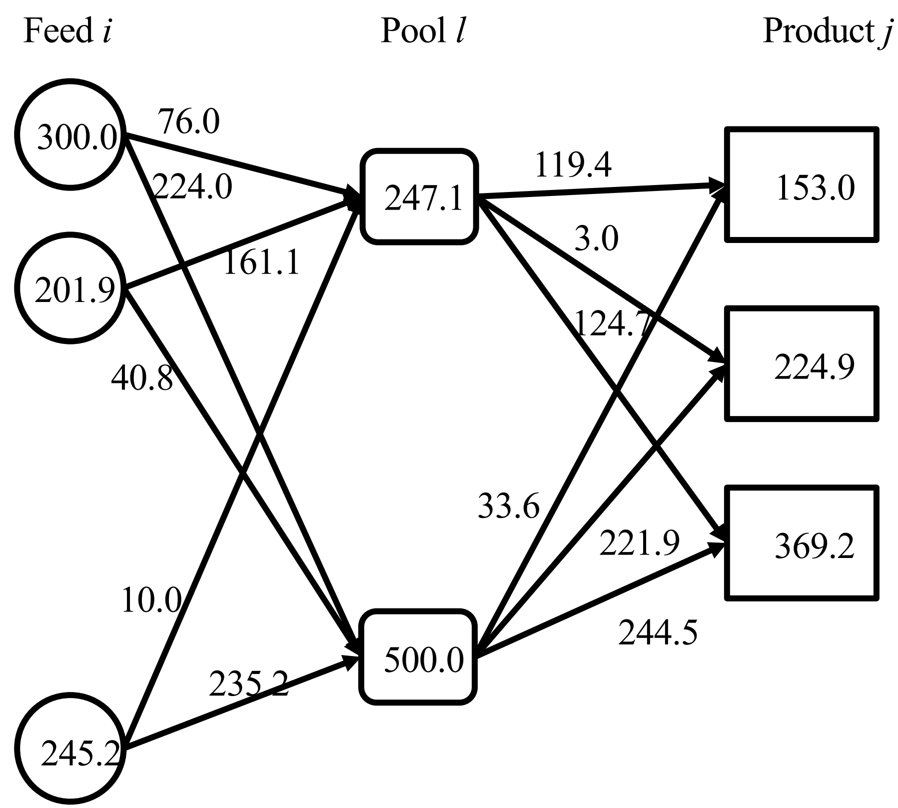
Profit = 2785.87

Stochastic Pooling Problem with Contract Selection

Case study – 3 scenario problem

Second stage decisions—high demand

Contracts	fixed	Discount after certain amount	Bulk discount
Feed 1		300.0	
Feed 2			201.9
Feed 5			245.2



Profit = 4419.94

Stochastic Pooling Problem with Contract Selection

#Variables: 9 binary, 9 continuous in stage 1. **32 binary,** 112 continuous in stage 2 per scenario.
#Constraints: 18 linear stage 1. 116 linear, **22 nonlinear** stage 2 per scenario.

Deterministic Equivalent

#Scenarios	3	9	27
BARON 18.5.8	5/0.1	3005/0.1	10^4/8.7
ANTIGONE 1.1	16/0.1	251/0.1	10^4/1.4
SCIP 5.0	10^4/54.4	10^4/100.0	10^4/100.0

Walltime/gap

Decomposition Algorithms

#Scenarios	3	9	27
GBD (with cuts)+L	152/0.1 1	502/0.1 1	2113/0.1 1
GBD+L	10^4/0.1 381	10^4/0.8 39	10^4/1.3 7
LD	10^4/0.2 363	10^4/7.1 43	10^4/12.2 9

Closes the gap at the root node

LD:Lagrangean Decomposition
L:Lagrangean cuts
GBD:Generalized Benders Decomposition

Walltime/gap
#nodes

Conclusions and Future Work

- **Cutting planes** can potentially reduce the number of **nodes** in the proposed **Generalized Benders decomposition-based branch and cut** algorithm
- **Heuristics** on when to add the cutting planes should be proposed in the future

Li, Can, and Ignacio E. Grossmann. "A generalized Benders decomposition-based branch and cut algorithm for two-stage stochastic programs with nonconvex constraints and mixed-binary first and second stage variables." Journal of Global Optimization 75.2 (2019): 247-272.

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